$$T = \frac{B}{(4\pi k\rho c)^{\frac{1}{2}}} \int_0^p \frac{t'^i \exp\{-[\rho c y^2/4k(t-t')]\}}{(t-t')^{\frac{1}{2}}} \mathrm{d}t', \qquad t > p, \quad i > -1. \tag{14b}$$

For $p/t \geqslant 1$, the given solutions [equations (13a) and (14a)] are equivalent to the classical exponential integral solution for a continuous plane source. For $p/t < 1$, the solution is that of a continuous source which then decays when the source is shut off.

For the case where $i = 0$ (constant heat generation), the integrated expression of equation (14a) is given on page 263 of [1]. Equations (14a) and (14b) can also be integrated by a term by term integration of the series expansion of the integrand. The resulting expression for equation (14a), for $t \leqslant p$, is found to be easier to use than equation (13a) because the former converges faster. For the same reason, equation (13b) for $t > p$, is more convenient to use than equation (14b). Furthermore, the involved numerical integration for various values of p as given by equation (14b) is avoided by using the more accessible parametric representation of equation (13b). The results of the integration of the latter at various values of p/t for the important case $i = 0$, is shown in Fig. 1 and for the case of ramp heat generation ($i = 1$), is shown in Fig. 2. A comparison of the two graphs shows that for the same period of heat generation and for the same amount of heat generated ($Bp^2/2 = Qp$), the temperature near the origin is higher for the case of ramp heat generation. For larger values of $t(t > 10\,p)$, the graphs show no appreciable difference between the corresponding temperature profiles.

ACKNOWLEDGEMENT

The full support of this work by NASA under Grant No. NGR 14-004-008 (041) from the Space Nuclear Propulsion Office is gratefully acknowledged.

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Int. J. Heat Mass Transfer. Vol. 15, pp. 164–169. Pergamon Press 1972. Printed in Great Britain

A FORCED CONVECTIVE HEAT TRANSFER INCLUDING DISSIPATION FUNCTION AND COMPRESSION WORK FOR A CLASS OF NONCIRCULAR DUCTS

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(*Received* 22 *March* 1971)

NOMENCLATURE

a,	physical length, introduced in (9) and (16) with different meanings;
A,	area of cross-section of duct;
b,	physical length, introduced in (16);
c_p,	specific heat referred to mass and constant pressure;
c_1, c_2, c_4,	$(1/\mu)(\mathrm{d}p/\mathrm{d}z^*)$, $\rho c_p\bar{\tau}/k$, $c_1 \cdot c_2$ respectively;
c_3, c_3^*,	Q/k, c_3/c_4a^2 respectively;
D,	domain of cross-section of duct;
De,	hydraulic diameter, $4A/s$;
$E(\sqrt{1-\lambda^2})$,	complete elliptic integral of second kind;
$f(z), g(z)$,	functions of z, introduced in (3) and (5) respectively;
h,	heat transfer coefficient;
k,	thermal conductivity coefficient;
L,	boundary of D;
Nu,	Nusselt number;
p,	pressure where $\mathrm{d}p/\mathrm{d}z^*$ is a negative constant;
q,	heat transfer rate at the wall;
Q,	intensity of heat source (or sink) distribution in the fluid medium;

s,	linear measure of L;
t,	actual local temperature;
T,	temperature difference, $t - t_w$;
u,	local velocity in the positive direction of z^*;
x, y, z^*,	Cartesian coordinates, x and y are contained in cross-sectional plane, z^*-axis is parallel to duct axis and direction of u;
z,	complex variable, $x + iy$, where $i = \sqrt{-1}$;
μ,	viscosity coefficient;
ρ,	density;
η, η^*,	μ/k, $(dp/dz^*)/\rho c_p \tau$;
Φ,	eccentric angle, introduced in (28);
τ,	$\partial t/\partial z^*$ ($= dt_w/dz^*$), which is a positive or a negative constant;
∇^2,	Laplacian operator, $\partial^2/\partial x^2 + \partial^2/\partial y^2$.
	bar, denotes conjugate complex of, e.g. $\bar{z}$ ($= x - iy$), $\bar{f}(\bar{z})$ and $\bar{g}(\bar{z})$ are conjugate complex of $z, f(z)$ and $g(z)$ respectively.

Subscripts

l,	local value;
m,	mean value over D;
M,	mixed-mean value over D;
0,	over-all value over L;
w,	value at the wall.

1. INTRODUCTION

CONSIDER a laminar, steady, constant property, forced convective and fully developed (both hydrodynamically and thermally) rectilinear flow of a Newtonian fluid inside a straight noncircular duct of constant area of cross-section. Consider the presence of some axially constant heat source (or sink) distributions in the wall and fluid materials. Let either of the following two thermal boundary conditions be prescribed. (i) Peripherally constant wall temperature, which varies linearly in the axial direction. (ii) Local normal temperature gradient at the wall; which remains constant in the axial direction and varies in the peripheral direction.

Since this problem is a basic and fundamental problem in the heat transfer science and important from engineering view point, a large number of research workers have studied it. We have studied the problem in a most general form in [1] and [2] for the thermal boundary conditions (i) and (ii) respectively. It is noteworthy that the studies [1] and [2] are the first to account for viscous dissipation and work of compression for the problem.

Taking the problem with thermal boundary condition (i), which has been analysed in [1] by means of conformal mapping technique, we propose to give here an analysis for those classes of the noncircular ducts L and the heat source (or sink) distribution Q for which closed form exact solutions are obtainable directly from the equations of L without using conformal mapping technique. The governing equations, with no-slip flow conditions at the wall, are

$$\nabla^2 u = c_1, \qquad u = 0 \quad \text{on} \quad L \tag{1}$$

$$\rho c_p \tau u = k\nabla^2 t + Q + \mu\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} + u\frac{dp}{dz^*}, \quad t = t_w(z^*) \quad \text{on} \quad L. \tag{2}$$

Some of the members of these classes (e.g. Q = constant, equilateral triangular duct, elliptic duct, etc.) are quite important in engineering sciences. However, the task of deducing the closed form exact solutions for the present class of L from [1] (where the general solutions have been given in terms of power series) would be difficult.

2. COMPARISON WITH L. N. TAO's WORK [3]

A similar kind of analysis has been given by Tao in [3]. However, the present study differs mainly from [3] in the following aspects: (i) Tao has not considered the phenomena, namely, viscous dissipation and work of compression in his analysis, whereas we are considering them here. In fact, these phenomena are quite significant as discussed in [1, 2]. Moreover, one may remark that in order to achieve a precision in the analysis, one should always consider these phenomena (even if they are relatively unimportant) in those cases at least where mathematical procedure does not complicate. However, the present case is one of those (ii) Tao's mathematical procedure results into solutions of biharmonic and Laplace equations and evaluation of velocity field from temperature field, whereas the present mathematical procedure will be resulting into solutions of Laplace equation and direct evaluation of velocity field.

Thus the present study may be regarded as a generalization of Tao [3] and gives a procedure which is obviously direct and mathematically simpler as compared to that in Tao [3].

3. ANALYSIS

In brief, our procedure and results are as follows:

Setting $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$ in (1) and reducing it to homogeneous form, we find that the solution of (1) is directly obtainable from the equations of L when they are expressible as

$$z\bar{z} = f(z) + \bar{f}(\bar{z}), \tag{3}$$

where $\bar{f}(\bar{z})$ denotes the conjugate complex of $f(z)$.

The solution is

$$u = \frac{c_1}{4}\{z\bar{z} - f(z) - \bar{f}(\bar{z})\}. \tag{4}$$

Now, introducing the complex variables and the velocity field (4) into (2) and performing some simple mathematical manipulations, we find that the solution for temperature is also obtainable directly from the equations of L, i.e. (3), provided (3) is expressible as

$$(z\bar{z})^2 = 4\{\bar{z}\int f(z)dz + z\int \bar{f}(\bar{z})\,d\bar{z}\} + g(z) + \bar{g}(\bar{z}), \qquad (5)$$

and $Q(x, y)$ has a special form (e.g. Q = constant). The result for the case of Q = constant (which is important in engineering science) is as follows.

$$T = t - t_w = \frac{c_4}{64}[(z\bar{z})^2 - 4\{\bar{z}\int f(z)dz + z\int \bar{f}(\bar{z})d\bar{z}\} - g(z) - \bar{g}(\bar{z})] + \frac{c_3}{4}\{f(z) + \bar{f}(\bar{z}) - z\bar{z}\} - \frac{\eta c_1{}^2}{32}\{f(z) + \bar{f}(\bar{z}) - z\bar{z}\}^2. \qquad (6)$$

The overall wall heat flux can directly be evaluated by applying Gauss Divergence Theorem to (2). The result for the case of Q = constant is

$$q_0 = Ak(c_2u_m - c_3). \qquad (7)$$

Now, the heat transfer coefficient h_0 and the Nusselt number Nu_0 can in general be calculated in view of the following definitions.

$$h_0 = -q_0/ST_M, \qquad Nu_0 = h_0De/k. \qquad (8)$$

Examples

Now we take some examples of L which satisfy (3) and (5), and offer the applications of the previous derivations to them.

Equilateral triangular duct. Let the equations of L for an equilateral triangular duct of side $2a\sqrt{3}$ be, [3].

$$x - a = 0, \quad x - y\sqrt{3} + 2a = 0, \quad x + y\sqrt{3} + 2a = 0. \quad (9)$$

Our results for this duct are as follows:

$$u = \frac{c_1}{12a}u_1, \qquad u_1 = (x - a\{(x + 2a)^2 - 3y^2\}.$$

$$u_m = -\frac{3}{20}c_1a^2, \qquad (10)$$

$$T = \frac{C_4}{192a}u_1[x^2 + y^2 - 4a^2 - 16a^2c_3^* - \eta^*(2/3\,a)u_1], \quad (11)$$

$$T_m = \frac{9}{280}c_4a^4(1 + \frac{14}{3}c_3^* - \frac{1}{2}\eta^*), \qquad (12)$$

$$T_M = \frac{27}{560}c_4a^4B_1, \qquad B_1 = \left(1 + \frac{40}{9}c_3^* - \frac{6}{11}\eta^*\right), \qquad (13)$$

$$q_0 = -\frac{1}{20\sqrt{3}}c_4kA^2B_2, \qquad B_2 = \left(1 + \frac{20}{3}c_3^*\right), \qquad (14)$$

$$Nu_0 = (28/9)(B_2/B_1). \qquad (15)$$

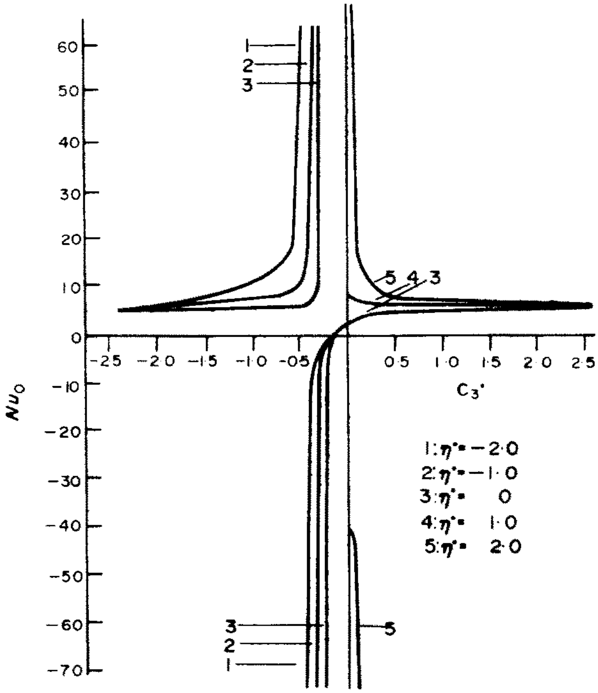

FIG. 1. Equilateral triangular duct: Nusselt number.

A graphical picture of Nu_0 [given by (15)] is given in Fig. 1.

Elliptic duct. Let the equation of L for an elliptic duct be, [3].

$$(x^2/a^2) + (y^2/b^2) = 1, \qquad a > b. \qquad (16)$$

The results for this duct are:

$$u = \frac{c_1}{2}u_2, \qquad u_2 = (b^2x^2 + a^2y^2 - a^2b^2)/(a^2 + b^2).$$

$$u_m = -\frac{1}{4}c_1a^2d. \qquad (17)$$

$$T = \frac{c_4}{32}u_2[(A_1 - 2A_2)x^2 + (A_1 + 2A_2)y^2 + 2A_1a^2d - 8a^2d - 16a^2c_3^* - \eta^*4u_2], \qquad (18)$$

$$T_m = \frac{1}{4}c_4a^4d\left(\frac{1}{3}d + c_3^* - \frac{1}{6}d\eta^*\right) \qquad (19)$$

$$T_M = \frac{c_4a^4}{144}B_3d,$$

$$B_3 = \frac{(17 + 98\lambda^2 + 17\lambda^4)d}{1 + 6\lambda^2 + \lambda^4} + 48c_3^* - 9\eta^*d. \qquad (20)$$

$$q_0 = -\frac{c_4kA^2}{4\pi\lambda^2}B_4, \qquad B_4 = d + 4c_3^*. \qquad (21)$$

$$Nu_0 = 9\left(\frac{1}{d}\right)\left\{\frac{\pi}{E(\sqrt{1-\lambda^2})}\right\}^2 [B_4/B_3], \quad (22)$$

where

$$d = \frac{\lambda^2}{1+\lambda^2}, \quad \lambda = \frac{b}{a}, \quad A_1 = \frac{2(1+10\lambda^2+\lambda^4)}{3(1+6\lambda^2+\lambda^4)},$$

$$A_2 = \frac{1-\lambda^4}{3(1+6\lambda^2+\lambda^4)} \quad (23)$$

A graphical picture of Nu_0 [given by (22)] is given in Fig. 2.

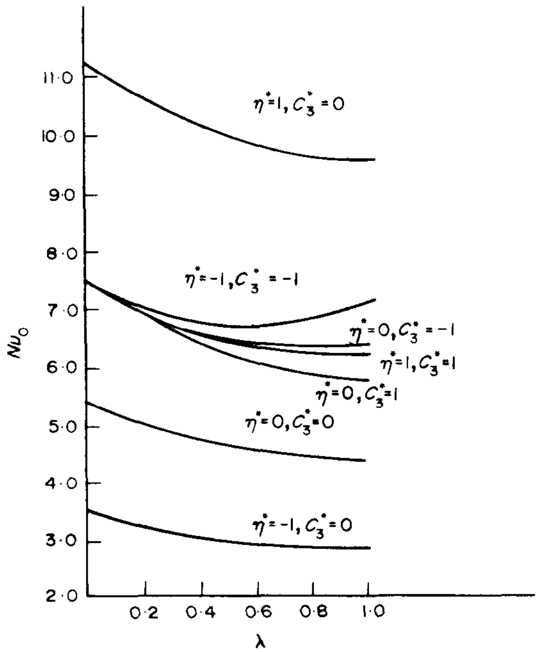

FIG. 2. Elliptic duct: Nusselt number.

4. CLOSURE

(i) In the foregoing, η (or η^*) stands for viscous dissipation and work of compression. Therefore, the derivations of Tao [3] are deducible by setting $\eta = 0$ ($\eta^* = 0$) into the present derivations.

(ii) In connection with, the derivation of the general solutions (4) and (6) (for the velocity and temperature fields respectively), we make the following statement: "For all members of the class of noncircular ducts defined by (3), the equation (3) cannot be transformed to the form (5) by means of any algebraic manipulations". The proof of this statement is furnished by the following example. Consider a noncircular duct whose equations are

$$x^2 + y^2 = R_1^2, \quad x^2 + y^2 = 2R_2x \ (R_1 < R_2). \quad (24)$$

Under the use of complex variables and some algebraic manipulations, equations (24) go to

$$z\bar{z} = \left[R_2\left(z - \frac{R_1^2}{z}\right) + \frac{1}{2}R_1^2\right] + \left[R_2\left(\bar{z} - \frac{R_1^2}{\bar{z}}\right) + \frac{1}{2}R_1^2\right] \quad (25)$$

which belongs to (3). It can easily be seen that (25) does not go to the form (5) under any algebraic manipulations. Thus, the solution (6) for T has been obtained for a subclass of the class (3). The statement and the related discussion, given above, have not been given in Tao [3].

(iii) The subject matter of local wall heat flux distribution, q_l, for noncircular ducts is quite significant as far as the laminar flow is concerned and is of great interest in engineering design. In order to obtain q_l from (6), we find that the general formula is

$$q_l = k\frac{\partial T}{\partial n} = k\left(e^{i\gamma}\frac{\partial T}{\partial z} + e^{-i\gamma}\frac{\partial T}{\partial \bar{z}}\right)L, \quad (26)$$

where n and γ denote outward drawn normal direction at L and the angle between the directions of n and positive x-axis respectively. For the foregoing equilateral triangular and elliptic ducts, we find, respectively,

$$\frac{q_l}{kc_4a^3} = \frac{1}{32}(X^2 + Y^2 - 4 - 16c_3^*)\,[\cos\gamma[X + \tfrac{1}{2}(X^2 - Y^2)] - \sin\gamma(XY - Y)], \quad (27)$$

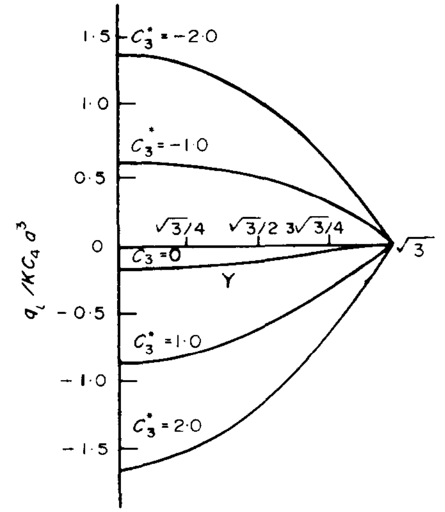

FIG. 3. Equilateral triangular duct: Local wall heat flux on one half of any side.

where $X = x/a$ and $Y = y/a$,

$$\frac{q_l}{kc_4a^3} = \frac{d}{3\lambda}\sqrt{(\sin^2\Phi + \lambda^2\cos^2\Phi)}\left[\frac{d}{1 + 6\lambda^2 + \lambda^4}[(1 - \lambda^2) \times (\cos^2\Phi - \lambda^2\sin^2\Phi) - (1 + 4\lambda^2 + \lambda^4)] - 3c_3^*\right]. \quad (28)$$

Examining equations (27) and (28), we find that the combined effect of viscous dissipation and work of compression on q_l is zero. However, this is not true for T, T_m, T_M and Nu_0 (as can be verified from the derivation given here). The subject matter of local wall heat flux distribution has been omitted in Tao [3]. Therefore, in connection with (27) and (28), we provide Fig. 3 and Fig. 4 also.

(iv) Let the percentage errors due to the omission of viscous dissipation and work of compression be denoted by ε_1, ε_2 and ε_3 for T_m, T_M and Nu_0 respectively. Then, in the case of $Q = 0$, we have, for equilateral triangular duct,

η^*	ε_1	ε_2	ε_3
1·0	−100·00	−120·00	54·5454
0·1	−5·2632	−5·77	5·4545

Looking into the table, we find that the quantitative effects are considerable when the magnitude of η^* is greater than 0·1.

(v) The case of circular tube of radius, a, is deducible from that of elliptic duct by setting $b = a$ in (16)–(23) and (28). Similarly, the case of flat duct, composed of two infinite parallel flat plates with gap $2b$, can be deduced from that of elliptic duct by letting a approach infinity and b remain finite.

(vi) The question of giving an analysis of the present kind for the thermal boundary condition (ii) is still open. However, we have tried but no success has been met.

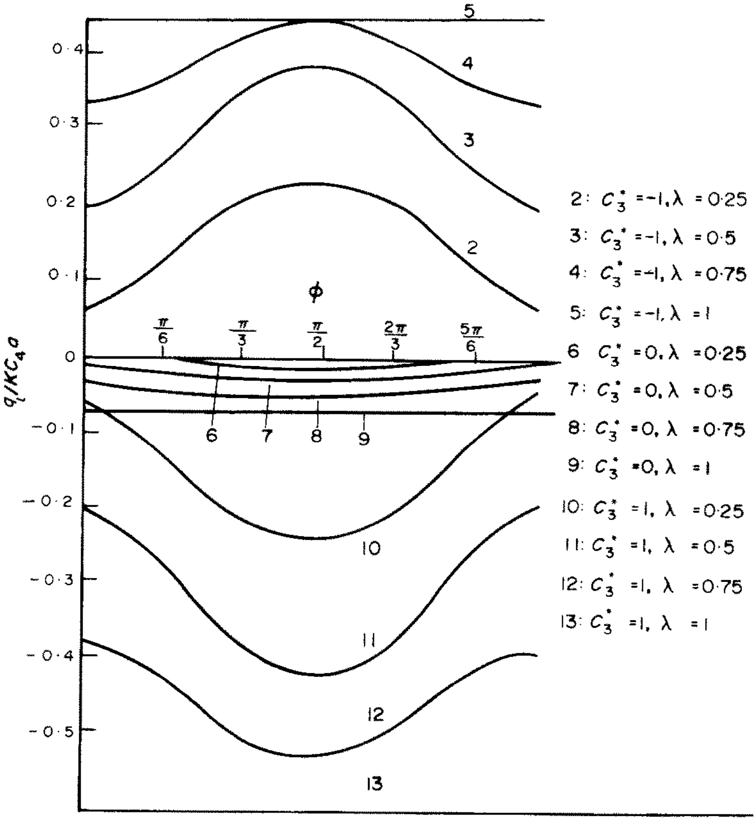

FIG. 4. Elliptic duct: Local wall heat flux on one half by major axis.

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Int. J. Heat Mass Transfer. Vol. 15, pp. 169–172. Pergamon Press 1972. Printed in Great Britain

USE OF PARALLEL POLARIZED RADIATION IN DETERMINATIONS OF OPTICAL CONSTANTS AND THICKNESS OF FILMS

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(*Received* 10 *March* 1971)

INTRODUCTION

A KNOWLEDGE of the optical constants of dielectric coatings and films is relevant to the calculation of radiation properties such as reflectivity and absorptivity which, in turn, are employed to determine radiative heat transfer. Several methods are developed in [1] for deducing optical constants and film thickness from monochromatic, specular reflectivity measurements at varying angle of incidence. The methods described therein were formulated for perpendicular polarized radiation. The purpose of the present note is to extend the formulation to accommodate parallel polarized radiation.

The physical situation under study is pictured schematically in the inset of Fig. 1. Consideration is given both to transparent films and to slightly absorbing films. Although the former correspond to the conventional analytical model for thin dielectric films, the experiments of [1] indicate that the latter may provide a closer representation to reality. To facilitate a concise presentation, liberal use will be made of the findings of [1].

TRANSPARENT FILMS

Consider first a transparent film ($\hat{n}_2 = n_2$) on an absorbing substrate [$\hat{n}_3 = n_3(1 + i\kappa_3)$]. For monochromatic, parallel polarized radiation incident under an angle θ_1, the Fresnel reflection coefficients at the interfaces 1–2 and 2–3, r_{12} and ρ_{23}, respectively, are

$$r_{12} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2},$$

$$\rho_{23}\, e^{i\phi_{23}} = \frac{\hat{n}_3 \cos\theta_2 - n_2 \cos\theta_3}{\hat{n}_3 \cos\theta_2 + n_2 \cos\theta_3} \quad (1)$$

in which ϕ_{23} is the phase shift at the interface 2–3. The angles θ_1, θ_2 and θ_3 are related by Snell's law, according to which $n_1 \sin\theta_1 = n_2 \sin\theta_2 = \hat{n}_3 \sin\theta_3$.

At the Brewster angle defined by $\theta_1 = \theta_1^* = \tan^{-1}(n_2/n_1)$, it can be shown that $r_{12} = 0$. Furthermore, for situations characterized by $n_1 \cong 1$ and $n_2 > 1$, it follows that $r_{12} > 0$ for $\theta_1 < \theta_1^*$ and $r_{12} < 0$ for $\theta_1 > \theta_1^*$. This behaviour of r_{12} introduces interesting differences in the present development relative to that for perpendicular polarized radiation.

In terms of the foregoing quantities, the monochromatic specular reflectivity is expressible as

$$R = \frac{r_{12}^2 + \rho_{23}^2 + 2r_{12}\rho_{23}\cos(\phi_{23} + 2\beta)}{1 + r_{12}^2\rho_{23}^2 + 2r_{12}\rho_{23}\cos(\phi_{23} + 2\beta)},$$

$$\beta = \left(\frac{2\pi}{\lambda_0}\right) n_2 h \cos\theta_2 \quad (2)$$

where λ_0 is the wavelength (in vacuum) of the incident